



Lecture 1

Number Systems and Codes

1



Agenda

- Number Systems
- Arithmetic
- Base Conversions
- Signed Number Representation
- Computer Code

2



Number Systems

- A number system consists of an ordered set of symbols, called *digits*
- Relation defined for addition (+), subtraction (-), multiplication (x) and division (\div)
- The radix (r), or bases, is the total number of digits allowed in the number system

3



Number Systems

- In computer programming and digital system design : decimal ($r=10$), binary ($r=2$), octal ($r=8$) and hexadecimal ($r =16$)
- Consist of 2 part : Integer part and fractional part, Separated by a radix point (.)

4



Number Systems

- ₪134.50 in the bank → cash :
 - 100 = 1, 10 = 3, 1 = 4, 0.50 = 1
- The position of each digit indicate its relative weight or significance

5



Number Systems

- Positional Notation
 - In general, a positive number N can be written in position notation as :
$$N = (a_{n-1} a_{n-2} \dots a_1 a_0 . a_{-1} a_{-2} \dots a_{-m})_r$$
 - . = radix point separating the integer and fractional digit
 - r = radix or base of number system being used
 - n = number of integer digit to the left of radix point
 - m = number of fractional digits to the right of the radix point
 - a_i = integer digit i when $n-1 \geq i \geq 0$
 - a_i = fractional digit i when $-1 \geq i \geq -m$
 - a_{n-1} = most significant digit
 - a_{-m} = least significant digit

6



Number Systems

- Polynomial Notation

- $(134.50)_{10} = 1 \times 100 + 3 \times 10 + 4 \times 1 + 5 \times 0.1$
 $= 1 \times 10^2 + 3 \times 10^1 + 4 \times 10^0 + 5 \times 10^{-1}$

- Number N of radix r may be written as a polynomial in the form

$$N = \sum_{i=-m}^{n-1} a_i r^i$$

- $r=10$, $a_2=1$, $a_1=3$, $a_0=4$, $a_{-1}=5$ and $a_i=0$ for $i \geq 3$ and for $i \leq -2$

7



Number Systems

- Commonly Used Number Systems

- Digital System using 2 state device.
- Representing numbers in digital is 0 , 1 call *bits*.
- Bits can be stored in a two-state storage device called *latch*.
- Binary numbers of length n can be stored in an n -bit long devices known as a *register*.

8

Number Systems

- Commonly Used Number Systems

TABLE 1.1 IMPORTANT NUMBER SYSTEMS

Name	Decimal	Binary	Octal	Hexadecimal
Radix	10	2	8	16
Digits	0, 1, 2, 3, 4, 5, 6, 7, 8, 9	0, 1	0, 1, 2, 3, 4, 5, 6, 7	0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F
First positive integers	0	0	0	0
	1	1	1	1
	2	10	2	2
	3	11	3	3
	4	100	4	4
	5	101	5	5
	6	110	6	6
	7	111	7	7
	8	1000	10	8
	9	1001	11	9
	10	1010	12	A
	11	1011	13	B
	12	1100	14	C
	13	1101	15	D
	14	1110	16	E
	15	1111	17	F
	16	10000	20	10

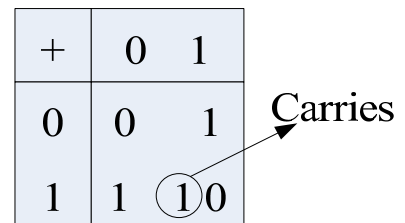
9

Arithmetic

- Binary Arithmetic

- Addition

1 1 1 1 1 1	<i>Carries</i>
1 1 1 1 0 1	Augend
1 0 1 1 1	+ Addend
1 0 1 0 1 0 0	Sum



10

Arithmetic

Add the four numbers $(101101)_2$, $(110101)_2$, $(001101)_2$, and $(010001)_2$.

$\begin{array}{r} 1111 \\ + 101101 \\ \hline 1100010 \end{array}$	$\begin{array}{r} 001101 \\ + 010001 \\ \hline 011110 \end{array}$	$\begin{array}{r} 10 \quad 10 \quad 10 \quad 10 \quad 1 \quad 10 \quad \text{Carries} \\ \quad 1 \quad 0 \quad 1 \quad 1 \quad 0 \quad 1 \\ \quad 1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \\ \quad 0 \quad 0 \quad 1 \quad 1 \quad 0 \quad 1 \\ + \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 1 \\ \hline 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \text{Sum} \end{array}$
---	--	---

$\begin{array}{r} 111111 \\ + 1100010 \\ \hline 10000000 \end{array}$

11

Arithmetic

■ Subtraction

- $1 - 0 = 1$
- $1 - 1 = 0$
- $0 - 0 = 0$
- $0 - 1 = 1$

(with a borrow 1)

Subtract $(10111)_2$ from $(1001101)_2$.

	6	5	4	3	2	1	0	Column
		1			10			Borrows
	0	1	0	10	0	0	10	Borrows
	1	0	0	1	1	0	1	Minuend
-			1	0	1	1	1	Subtrahend
	1	1	0	1	1	0		Difference

12



Base Conversions

- Conversion Methods

- Series Substitution :

Convert $(10100)_2$ to base 10.

$$\begin{aligned}N &= 1 \times 2^4 + 0 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 0 \times 2^0 \\ &= (16)_{10} + 0 + (4)_{10} + 0 + 0 \\ &= (20)_{10}\end{aligned}$$

Convert $(274)_8$ to base 10.

$$\begin{aligned}N &= 2 \times 8^2 + 7 \times 8^1 + 4 \times 8^0 \\ &= (128)_{10} + (56)_{10} + (4)_{10} \\ &= (188)_{10}\end{aligned}$$

19



Base Conversions

- Series Substitution :

Convert $(AF3.15)_{16}$ to base 10.

$$\begin{aligned}N &= A \times 16^2 + F \times 16^1 + 3 \times 16^0 + 1 \times 16^{-1} + 5 \times 16^{-2} \\ &= 10_{10} \times 256_{10} + 15_{10} \times 16_{10} + 3_{10} \times 1_{10} \\ &\quad + 1_{10} \times 0.0625_{10} + 5_{10} \times 0.00390625_{10} \\ &= 2560_{10} + 240_{10} + 3_{10} + 0.0625_{10} + 0.01953125_{10} \\ &= (2803.08203125)_{10}\end{aligned}$$

20

Base Conversions

- Radix Divide Method

Convert $(234)_{10}$ to base 8.

We solve this problem by repeatedly dividing integer $(234)_{10}$, that is $(N)_A$, by 8, that is $(B)_A$, until the quotient is 0.

$$\begin{array}{r} 29 \\ 8 \overline{) 234} \\ \underline{16} \\ 74 \\ \underline{72} \\ 2 = b_0 \end{array}$$

$$\begin{array}{r} 3 \\ 8 \overline{) 29} \\ \underline{24} \\ 5 = b_1 \end{array}$$

$$\begin{array}{r} 0 \\ 8 \overline{) 3} \\ \underline{0} \\ 3 = b_2 \end{array}$$

$$\begin{array}{r} 234 \\ 8 \overline{) 234} \\ \underline{16} \\ 74 \\ \underline{72} \\ 2 \\ 8 \overline{) 74} \\ \underline{64} \\ 10 \\ \underline{8} \\ 2 \\ 8 \overline{) 10} \\ \underline{8} \\ 2 \\ 8 \overline{) 2} \\ \underline{0} \\ 2 \end{array} \begin{array}{l} \uparrow \\ \text{LSB} \\ \text{MSB} \end{array}$$

21

Base Conversions

- Radix Divide Method

Convert $(234)_{10}$ to base 16.

$$\begin{array}{r} 14 \\ 16 \overline{) 234} \\ \underline{16} \\ 74 \\ \underline{64} \\ 10 = (A)_{16} = b_0 \end{array}$$

$$\begin{array}{r} 0 \\ 16 \overline{) 14} \\ \underline{0} \\ 14 = (E)_{16} = b_1 \end{array}$$

Hence, $(234)_{10} = (EA)_{16}$. In the shorthand notation;

$$\begin{array}{r} 234 \\ 16 \overline{) 234} \\ \underline{16} \\ 74 \\ \underline{64} \\ 10 = (A)_{16} \\ 16 \overline{) 74} \\ \underline{64} \\ 10 \\ \underline{8} \\ 2 \\ 16 \overline{) 10} \\ \underline{8} \\ 2 \\ 16 \overline{) 2} \\ \underline{0} \\ 2 \end{array} \begin{array}{l} \uparrow \\ \text{MSB} \\ \text{LSB} \end{array}$$

22

Base Conversions

- Radix Multiply Method

Convert $(0.1285)_{10}$ to base 8.

0.1285	0.0280	0.2240	0.7920
$\times 8$	$\times 8$	$\times 8$	$\times 8$
1.0280	0.2240	1.7920	6.3360
↑	↑	↑	↑
b_{-1}	b_{-2}	b_{-3}	b_{-4}

0.3360	0.6880	0.5040	0.0320
$\times 8$	$\times 8$	$\times 8$	$\times 8$
2.6880	5.5040	4.0320	0.2560
↑	↑	↑	↑
b_{-5}	b_{-6}	b_{-7}	b_{-8}

MSD		$1.656250 \leftarrow 0.828125 \times 2$
		$1.312500 \leftarrow 0.656250 \times 2$
		$0.625000 \leftarrow 0.312500 \times 2$
		$1.250000 \leftarrow 0.625000 \times 2$
		$0.500000 \leftarrow 0.250000 \times 2$
LSD		$1.000000 \leftarrow 0.500000 \times 2$

$0.828125_{10} = (0.110101)_2$

Thus

$0.1285_{10} = (0.10162540\dots)_8$

Base Conversions

- General Conversion Algorithm

Convert $(18.6)_9 = (?)_{11}$

$N_A = (18.6)_9$

a. Converting to base 10 via series substitution yields

$$\begin{aligned}
 N_{10} &= 1 \times 9^1 + 8 \times 9^0 + 6 \times 9^{-1} \\
 &= 9 + 8 + 0.666\dots \\
 &= (17.666\dots)_{10}
 \end{aligned}$$

b. Converting from base 10 to base 11 via radix divide produces

11		17	6	.	7.326	$\leftarrow 0.666 \times 11$
11		1	1	.	3.586	$\leftarrow 0.326 \times 11$
		0			6.446	$\leftarrow 0.586 \times 11$

Putting the integer and fraction parts together,

$N_{11} = (16.736\dots)_{11}$

Base Conversions

- Conversion between Base A and Base B
: When $B = A^k$

Convert $(1011011.1010111)_2$ to base 8.

Algorithm 1.3a can be applied where $B = 8 = 2^3 = A^k$. Therefore, three binary digits are grouped for each octal digit.

$$\begin{array}{ccccccc} \underbrace{001} & \underbrace{011} & \underbrace{011} & . & \underbrace{101} & \underbrace{011} & \underbrace{100} \\ 1 & 3 & 3 & & 5 & 3 & 4 \end{array}$$

$$1011011.1010111_2 = (133.534)_8$$

25

Base Conversions

Convert $(AF.16C)_{16}$ to base 8.

Since both 16 and 8 are powers of 2, Algorithm 1.3 can be applied twice as follows.

- Use Algorithm 1.3b to convert $(AF.16C)_{16}$ to base 2, since $16 = 2^4$. Each hexadecimal digit is replaced by four binary digits.

$$\begin{array}{ccccccc} \underbrace{A} & \underbrace{F} & \underbrace{1} & \underbrace{6} & \underbrace{C} \\ 10101111 & . & 0001 & 0110 & 1100 \end{array}$$

$$(AF.16C)_{16} = (10101111.000101101100)_2$$

- Use Algorithm 1.3a to convert the binary number to base 8.

$$\begin{array}{ccccccc} \underbrace{010} & \underbrace{101} & \underbrace{111} & . & \underbrace{000} & \underbrace{101} & \underbrace{101} & \underbrace{100} \\ 2 & 5 & 7 & & 0 & 5 & 5 & 4 \end{array}$$

Therefore:

$$(AF.16C)_{16} = (257.0554)_8$$

26



Signed Number Representation

- Sign Magnitude Numbers

A signed number $N = \pm(a_{n-1} \dots a_0.a_{-1} \dots a_{-m})_r$ may be written magnitude form as follows.

$$N = (sa_{n-1} \dots a_0.a_{-1} \dots a_{-m})_{rsm}$$

where $s = 0$ if N is positive and $s = r - 1$ if N is negative.

Determine the sign-magnitude code of $N = -(13)_{10}$ in binary ($r = 2$) and decimal ($r = 10$).

In binary:

$$\begin{aligned} N &= -(13)_{10} \\ &= -(1101)_2 \\ &= (1, 1101)_{2sm} \end{aligned}$$

In decimal:

$$\begin{aligned} N &= -(13)_{10} \\ &= (9, 13)_{10sm} \end{aligned}$$

where 9 is used to represent the negative sign for $r = 10$.

27



Signed Number Representation

- Complementary Number Systems

- Radix Complement

The radix complement $[N]_r$ of a number $(N)_r$:

$$[N]_r = r^n - (N)_r$$

r = base

n = Number of digit

28

Signed Number Representation

- Radix Complement

Determine the two's complement of $(N)_2 = (01100101)_2$.

From Eq. 1.8,

$$\begin{aligned} [N]_2 &= [01100101]_2 \\ &= 2^8 - (01100101)_2 \\ &= (100000000)_2 - (01100101)_2 \\ &= (10011011)_2. \end{aligned}$$

29

Signed Number Representation

Determine the two's complement of $(N)_2 = (11010100)_2$, and verify that it can be used to represent $-(N)_2$ by showing that $(N)_2 + [N]_2 = 0$.

First we determine the two's complement from Eq. 1.8:

$$\begin{aligned} [N]_2 &= [11010100]_2 \\ &= 2^8 - (11010100)_2 \\ &= (100000000)_2 - (11010100)_2 \\ &= (00101100)_2. \end{aligned}$$

$$\begin{array}{r} 1\ 1\ 0\ 1\ 0\ 1\ 0\ 0 \\ + 0\ 0\ 1\ 0\ 1\ 1\ 0\ 0 \\ \hline 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ \uparrow \\ \text{carry} \end{array}$$

30



Signed Number Representation

$$\begin{aligned} [(N)_2]_2 &= [-(N)_2]_2 \\ &= -(-(N)_2)_2 \\ &= (N)_2 \end{aligned}$$

$$\begin{aligned} [N]_2 &= [11010100]_2 \\ &= 2^8 - (11010100)_2 &= 2^8 - (00101100)_2 \\ &= (100000000)_2 - (11010100)_2 &= (100000000)_2 - (00101100)_2 \\ &= (00101100)_2 &= (11010100)_2 \end{aligned}$$

31



Signed Number Representation

Determine the two's complement of $(N)_2 = (10110)_2$ for $n = 8$.

From Eq. 1.8.

$$\begin{aligned} [N]_2 &= [10110]_2 \\ &= 2^8 - (10110)_2 \\ &= (100000000)_2 - (10110)_2 \\ &= (11101010)_2 \end{aligned}$$

Find the 10's complement of $(N)_{10} = (40960)_{10}$.

$$\begin{aligned} [N]_{10} &= [40960]_{10} \\ &= 10^5 - (40960)_{10} \\ &= (100000)_{10} - (40960)_{10} \\ &= (59040)_{10} \end{aligned}$$

32

Signed Number Representation

Find the two's complement of $N = (01100101)_2$.

$$\begin{array}{cccccccc}
 N & = & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
 & & & & & & & & & \downarrow & \text{first nonzero digit} \\
 [N]_2 & = & (1 & 0 & 0 & 1 & 1 & 0 & 1 & 1)_2
 \end{array}$$

Find the two's complement of $N = (11010100)_2$.

$$\begin{array}{cccccccc}
 N & = & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
 & & & & & & & & & \downarrow & \text{first nonzero digit} \\
 [N]_2 & = & (0 & 0 & 1 & 0 & 1 & 1 & 0 & 0)_2
 \end{array}$$

33

Signed Number Representation

Find the two's complement of $N = (10110)_2$ for $n = 8$.

$$\begin{array}{cccccccc}
 N & = & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
 & & & & & & & & & \downarrow & \text{first nonzero digit} \\
 [N]_2 & = & (1 & 1 & 1 & 0 & 1 & 0 & 1 & 0)_2
 \end{array}$$

Find the 10's complement of $(40960)_{10}$.

$$\begin{array}{cccccc}
 N & = & 4 & 0 & 9 & 6 & 0 \\
 & & & & & & \downarrow & \text{first nonzero digit} \\
 [N]_{10} & = & (5 & 9 & 0 & 4 & 0)_{10}
 \end{array}$$

34

Signed Number Representation

Find the two's complement of $N = (01100101)_2$.

$$\begin{array}{r}
 N = 01100101 \\
 \quad 10011010 \quad \text{Complement the bits} \\
 \quad \quad +1 \quad \quad \text{Add 1} \\
 [N]_2 = (10011011)_2
 \end{array}$$

Find the two's complement of $N = (11010100)_2$.

$$\begin{array}{r}
 N = 11010100 \\
 \quad 00101011 \quad \text{Complement the bits} \\
 \quad \quad +1 \quad \quad \text{Add 1} \\
 [N]_2 = (00101100)_2
 \end{array}$$

Find the 10's complement of $(40960)_{10}$.

$$\begin{array}{r}
 N = 40960 \\
 \quad 59039 \quad \text{Complement the digits} \\
 \quad \quad +1 \quad \quad \text{Add 1} \\
 [N]_{10} = (59040)_{10}
 \end{array}$$

35

Signed Number Representation

Given $(N)_2 = (1100101)_2$, determine the two's complement number system representations of $\pm(N)_2$ for $n = 8$.

By inspection,

$$+(N)_2 = (0, 1100101)_{2\text{cns}}$$

From Eq. 1.8,

$$\begin{aligned}
 -(N)_2 &= [\pm(N)_2]_2 \\
 &= [0, 1100101]_2 \\
 &= 2^8 - (0, 1100101)_2 \\
 &= (100000000)_2 - (0, 1100101)_2 \\
 &= (1, 0011011)_{2\text{cns}}
 \end{aligned}$$

36



Signed Number Representation

Find the two's complement number system representations of $\pm(110101)_2$ for $n = 8$.

By inspection,

$$+(110101)_2 = (0, 0110101)_{2cns}$$

From Eq. 1.8,

$$\begin{aligned} -(110101)_2 &= [110101]_2 \\ &= 2^8 - (110101)_2 \\ &= (10000000)_2 - (110101)_2 \\ &= (1, 1001011)_{2cns} \end{aligned}$$

37



Signed Number Representation

Determine the two's complement number system encoding of $-(13)_{10}$ for $n = 8$.

We begin by converting $(13)_{10}$ from decimal to binary.

$$+(13)_{10} = +(1101)_2 = (0, 0001101)_{2cns}$$

Next we compute the two's complement of $(0, 0001101)_{2cns}$

$-(13)_{10}$:

$$\begin{aligned} -(13)_{10} &= -(0, 0001101)_{2cns} \\ &= [0, 0001101]_2 \\ &= 2^8 - (0, 0001101)_2 \\ &= (1, 1110011)_{2cns} \end{aligned}$$

38



Signed Number Representation

Determine the decimal number represented by $N = (1,111010)_{2,ms}$.

From the sign bit, we see that N is a negative number. Therefore, we determine the magnitude of N (the corresponding positive value) by computing its two's complement.

$$\begin{aligned} N &= (1,111010)_{2,ms} \\ &= -[1,111010]_2 \\ &= -(2^8 - (1,111010)_2) \\ &= -(0,000110)_{2,ms} \\ &= -(6)_{10} \end{aligned}$$

where $(0,000110)_{2,ms} = +(6)_{10}$. Therefore, $(1,111010)_{2,ms}$ represents $-(6)_{10}$.